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Non-Abelian structures in compactifications of M-theory on seven-manifolds with $\mathrm{SU}(3)$ structure

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Abstract: We study M-theory compactified on a specific class of seven-dimensional manifolds with SU(3) structure. The manifolds can be viewed as a fibration of an arbitrary Calabi-Yau threefold over a circle, with a U-duality twist around the circle. In some cases we find that in the four dimensional low energy effective theory a (broken) non-Abelian gauge group appears. Furthermore, such compactifications are shown to be dual to previously analyzed compactifications of the heterotic string on $K3 \times T^2$, with background gauge field fluxes on the T^2.

Keywords: String Duality, M-Theory.

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Contents

1. Introduction

The study of possible space-time backgrounds of string theories has been an active field of research for almost 25 years. A specific subclass of backgrounds admit a geometrical interpretation in which the space-time manifold is the product space

$$M_d \times Y_{10-d}\,, \tag{1.1}$$

where M_d is an infinitely extended d-dimensional manifold with Minkowskian signature while Y_{10-d} is a $(10-d)$-dimensional compact manifold with Euclidean signature. In standard compactifications, Y is constrained to be a Calabi-Yau manifold whose holonomy controls the amount of unbroken supersymmetry present in the string background. More generally, one can turn on background fluxes for various p-form fields in the compact directions, and then Y is no longer constrained to be Ricci-flat. Such 'generalized' compactifications have been studied intensively in recent years [1].

It has been observed early on that these generalized compactifications can be discussed in terms of 'manifolds with G-structure' [2]. Such manifolds admit a globally defined spinor (or tensor) which is left invariant by the subgroup G of the structure group. Generically such manifolds have torsion and they can be characterized by a set of non-vanishing torsion classes [3, 4]. In string compactifications the number of invariant spinors is directly related to the number of supersymmetries present in the background. Calabi-Yau manifolds are a specific subclass of manifolds with G-structure where the torsion vanishes and the invariant spinor is covariantly constant with respect to the Levi-Civita connection.

String theories have the feature that their space-time backgrounds can be dual to each other. This is firmly established for dualities which hold in string perturbation theory. For example type IIA string theory compactified on a Calabi-Yau threefold Y coincides with type IIB string theory compactified on the mirror Calabi-Yau $\tilde{Y}$. For dualities which involve the dilaton (the string coupling) in a non-trivial way, so far there is only (strong) evidence for the validity of the duality. An example of such a duality is the heterotic string compactified on $K3 \times T^2$, which is believed to be identical to type IIA string theory compactified on a $K3$-fibred Calabi-Yau threefold [5, 6].

An interesting question is the fate of these dualities for generalized compactifications. It has been shown in refs. [7 – 10] that mirror symmetry continues to hold in the supergravity limit for compactifications on manifolds with SU(3) $\times$ SU(3) structure. On the other hand, for the heterotic-type II duality mentioned above only partial results have been obtained so far [11, 12]. More specifically it has been shown in [12] that in the supergravity limit a particular class of SU(3) $\times$ SU(3)-structure compactifications of type IIA is dual to the heterotic string compactified on $K3 \times T^2$ with a specific choice of background fluxes. However, for some of the fluxes which can be turned on in the heterotic theory no dual type IIA compactification could be identified. These are fluxes which result in a gauged $N = 2$ supergravity with vector multiplets carrying a non-Abelian charge.

Let us review this in a little more detail. At the level of the low-energy effective supergravity the heterotic — type II duality corresponds to a duality between $N = 2$ supergravities in $d = 4$. Such supergravities can be coupled to $N = 2$ vector-, hyper- and tensor-multiplets.[1] Turning on background fluxes or compactifying on manifolds with G-structure correspond to the deformation of an ungauged supergravity into a gauged or massive supergravity, with the flux and torsion playing the role of non-trivial gauge charges or mass parameters. For SU(3) $\times$ SU(3)-structure compactifications of type II it was shown in [7, 9, 10, 13 – 16] that only gauged supergravities with charged hypermultiplets or massive tensor multiplets appear. It was left as a puzzling feature of such compactifications that charged vector multiplets could not be obtained. On the other hand in the heterotic $K3 \times T^2$ compactification, non-Abelian gauge symmetries do appear precisely when fluxes on the T^2 are turned on [17].

[1] A tensor multiplet can be dualized to a hypermultiplet or a vector multiplet, depending on the mass of the tensor.

One of the goals of this paper is to resolve this puzzle. We find that non-Abelian gauge symmetries also appear on the type IIA side if instead of $\mathrm{SU}(3)\times\mathrm{SU}(3)$-structure compactifications of type II one considers compactifications of M-theory on 7-dimensional manifolds with SU(3) structure. We argue that these are the duals of the heterotic compactifications with fluxes on the T^2. These are the main results of our paper.

In this paper we do not attempt to work out the low-energy effective action for a generic 7-dimensional SU(3)-structure manifold, but instead focus on a very specific subclass of SU(3)-structure manifolds which lead to non-Abelian gauge symmetries. More specifically, we consider 7-dimensional manifolds which can be viewed as a non-trivial fibration of a Calabi-Yau threefold CY_3 over a circle S^1. Furthermore, we impose that only the second cohomology $H^{(1,1)}$ of CY_3 is twisted when going around the S^1, but that the third cohomology H^3 (which governs the hypermultiplet sector) is left unchanged. This constraint leads to a hypermultiplet sector which is entirely determined by $H^3(CY_3)$ and therefore can be safely neglected for the purpose of this paper. It is precisely the twisted $H^{(1,1)}$ cohomology which induces the non-Abelian structure into the theory. When CY_3 is $K3$-fibred and M-theory on CY_3 is dual to heterotic string theory on $K3\times S^1$, twists of this type provide the dual of the T^2 fluxes on the heterotic side.

Let us describe the twisting in slightly more detail. Consistency requires that after going around the circle which takes us from 5 to 4 dimensions, $H^{(1,1)}$ is rotated by an element of the U-duality group [18, 19] of the five-dimensional theory which corresponds to M-theory compactified on CY_3 or the dual heterotic theory on $K3\times S^1$. In this case we have $\Gamma(\mathbf{Z}) = \mathrm{SO}(1, h^{(1,1)}-2, \mathbf{Z})$, which on the heterotic side is just the T-duality group. On the M-theory side this symmetry exists precisely for the dual K3-fibred Calabi-Yau manifolds [6].

In four dimensions the result is the appearance of non-Abelian gauge symmetries as follows. The base of the K3-fibration is a $\mathbf{P_1}$ whose volume is identified with the four dimensional heterotic dilaton. Heterotic weak coupling corresponds to a large $\mathbf{P_1}$-base, and in that limit the low-energy limits of both theories have a $\mathrm{SO}(2, h^{(1,1)}-1, \mathbf{R})$ symmetry in four dimensions. In this case, we can describe the twisting in the language of four dimensional supergravity as gauging[2] an isometry inside $\mathrm{SO}(2, h^{(1,1)}-1, \mathbf{R})$. This makes the four-dimensional gauge transformations non-commuting (non-Abelian).

At a generic point in field space, the non-Abelian gauge symmetry is spontaneously broken, giving a mass to some of the gauge bosons. We find that on the M-theory side the gauge boson masses are inversely proportional to the radius of the M-theory circle. In order to consistently keep these gauge bosons in the low energy effective action we need to require that these masses are smaller than the Kaluza-Klein masses of the CY_3. This means that the M-theory circle has to be larger than the radii of the Calabi-Yau, which in turn forces us into the M-theory regime of type IIA string theory. This is the reason that the non-Abelian structure is not visible in $\mathrm{SU}(3)\times\mathrm{SU}(3)$ compactifications of type II theories.

Let us stress that the flux on the heterotic side, and similarly the non-trivial monodromy in the M-theory compactification, lead to a non-trivial potential on the moduli

[2]By gauging we mean that isometries of the scalar manifold are mixed into the gauge transformations, and not that new gauge fields are introduced.

space. We do not discuss here the stabilization of these moduli, which can be accomplished by adding additional ingredients. Instead, we just compute and compare the resulting low-energy effective actions, without attempting to solve their equations of motion.

The paper is organized as follows. In section 2 we discuss the Kaluza-Klein (KK) reduction of M-theory on a seven-dimensional manifold with SU(3) structure. As a warm-up, we first recall in section 2.1 the properties of the five-dimensional background corresponding to the reduction of M-theory on a Calabi-Yau threefold. This sets the stage for the specific S^1-fibration we consider in section 2.2. In section 2.3 we derive the low energy effective supergravity by a Kaluza-Klein reduction from 11 to 4 dimensions, paying special attention to the gauging of the vector multiplets. In section 2.4 we rewrite the effective action in a form which shows the consistency with $N = 2$ gauged supergravity. In section 2.5 we consider the specific case of a $K3$-fibred Calabi-Yau threefold which is the class of backgrounds dual to the heterotic string. Most of our explicit computations are done in the limit in which the scalar moduli space has a continuous isometry, as this makes the computations simpler; in section 2.6 we discuss what happens when we go away from this limit. In section 3 we turn to the heterotic string compactified on $K3 \times T^2$ and start by recalling a few generic properties of such backgrounds in section 3.1. We then compare the mass-scales in the dual backgrounds in section 3.2, showing the necessity to go to the M-theory regime on the type II side when we turn on heterotic fluxes on the T^2. In section 3.3 we argue that also the heterotic fluxes can be viewed as a monodromy in the T-duality group. In section 3.4 we then recall the heterotic effective action as computed in [17]. Finally, in section 3.5 we compare the effective actions on both sides and show that for a subset of torsion parameters they perfectly match. For the convenience of the reader we briefly recall the vector multiplet sector of (gauged) $N = 2$ supergravity in appendix A. Additional details of the vector multiplets in heterotic string compactifications are assembled in appendix B.

2. M-theory compactifications on manifolds with SU(3) structure

In this section we compactify M-theory on seven-dimensional manifolds with SU(3) structure. By construction this leads to an $N = 2$ supersymmetric effective theory in $d = 4$. However, as already explained in the introduction, we do not consider the most general manifolds with SU(3) structure but instead focus on a particular subclass of manifolds which lead to a low-energy supergravity with non-Abelian vector multiplets. For simplicity we further insist that the moduli space of the hypermultiplets coincides with that of a $CY_3 \times S^1$ compactification, where all scalars in hypermultiplets are gauge neutral. Thus, we do not pay attention to the hypermultiplets but only concentrate on the vector multiplet sector. The gaugings which appear in the hypermultiplet sector in general compactifications of M-theory on manifolds with SU(3) structure and the corresponding prepotentials were derived in [20, 21], but a detailed analysis in the vector multiplet sector of these compactifications was not considered so far.

We begin with a short review of the compactification of M-theory on six dimensional Calabi-Yau manifolds, and then proceed to the seven dimensional case.

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2.1 M-theory compactifications on Calabi-Yau threefolds

In order to set the stage let us briefly recall the structure of the five dimensional $N = 2$ supergravity[3] which arises from compactifying M-theory on Calabi-Yau threefolds. Our discussion is based on references [22 – 24] but since we are only interested in the vector multiplet sector we (largely) ignore the hypermultiplets in this section.

The bosonic spectrum of eleven-dimensional supergravity is particularly simple and consists only of the metric $\hat{G}_{MN}$ and a three-form potential $\hat{C}_3$. (We use hats ˆ in order to denote the eleven-dimensional quantities.) The eleven-dimensional action for these fields is given by (setting the eleven dimensional Newton's constant to one)

$$S_{11} = \frac{1}{2}\int \left[\hat{R} * 1 - \frac{1}{2}\hat{F}_4 \wedge *\hat{F}_4 - \frac{1}{6}\hat{F}_4 \wedge \hat{F}_4 \wedge \hat{C}_3\right] , \tag{2.1}$$

where $\hat{F}_4 = d\hat{C}_3$ is the field strength of the three-form potential $\hat{C}_3$.

The five-dimensional vector fields arise from expanding $\hat{C}_3$ in terms of harmonic $(1, 1)$-forms on the Calabi-Yau. More precisely we choose a basis ω_i of $H^{(1,1)}(CY_3)$ and expand according to

$$\hat{C}_3 = A^i \wedge \omega_i + \cdots , \qquad i = 1, \ldots , h^{(1,1)} , \tag{2.2}$$

where the ... indicate further terms corresponding to scalar fields in hypermultiplets. One of the vector fields A^i is identified with the graviphoton while the other $(h^{(1,1)} - 1)$ are members of vector multiplets. Their (bosonic) superpartners correspond to Kähler deformations of the Calabi-Yau metric. More precisely, one expands also the Kähler form J in terms of the basis ω_i

$$J = \nu^i \omega_i , \tag{2.3}$$

such that the ν^i parameterize the Kähler deformations. In the five-dimensional low energy effective theory the ν^i appear as scalar fields. However, one of the Kähler moduli, the overall volume $\mathcal{K}$, is not part of any vector multiplet but instead is a member of the universal hypermultiplet. The remaining $(h^{(1,1)} - 1)$ moduli are the scalar fields in vector multiplets.

Inserting (2.2) and (2.3) into (2.1) and integrating over the Calabi-Yau manifold results in the five-dimensional $N = 2$ effective action (for the bosonic fields that are not in hypermultiplets)[4]

$$S_5 = \int \left[\frac{1}{2}R_5 * \mathbf{1} - g^{(5)}_{\alpha\beta}\, d\varphi^\alpha \wedge *d\varphi^\beta - \frac{1}{4}g_{ij}\Big|_{\mathcal{K}=1} F^i \wedge *F^j - \frac{1}{12}\mathcal{K}_{ijk}F^i \wedge F^j \wedge A^k\right] , \tag{2.4}$$

where $F^i = dA^i$ and $\mathcal{K}_{ijk}$ are intersection numbers of the Calabi-Yau defined by the integral

$$\mathcal{K}_{ijk} = \int_{CY_3} \omega_i \wedge \omega_j \wedge \omega_k . \tag{2.5}$$

[3]By $N = 2$ we mean the minimal amount of supersymmetry possible in five dimensions, which reduces to $N = 2$ in four dimensions.

[4]Here we only give the final result and refer the reader for further details to [23, 24].

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To explain the other couplings in this action we need to be more explicit about the separation of the overall volume modulus $\mathcal{K}$ from the other Kähler moduli. Since the volume modulus is part of the universal hypermultiplet, it should not mix with the other quantities describing the vector multiplet moduli space. Therefore, all the terms in the vector multiplet action (2.4) are evaluated on a hypersurface of constant $\mathcal{K}$ which we choose as $\mathcal{K} = 1$. This is precisely the meaning of the matrix of gauge couplings $g_{ij}|_{\mathcal{K}=1}$ in the action (2.4), which is equal to the metric on the Kähler moduli space [25]

$$g_{ij} = \frac{1}{4\mathcal{K}} \int_{CY_3} \omega_i \wedge *\omega_j = -\frac{1}{4\mathcal{K}} \left(\mathcal{K}_{ij} - \frac{\mathcal{K}_i \mathcal{K}_j}{4\mathcal{K}} \right) , \tag{2.6}$$

evaluated on the hypersurface $\mathcal{K} = 1$.[5] Here the Calabi-Yau volume, $\mathcal{K}$, is defined as

$$\mathcal{K} = \frac{1}{6} \int_{CY_3} J \wedge J \wedge J = \frac{1}{6} \mathcal{K}_{ijk} \nu^i \nu^j \nu^k , \tag{2.7}$$

and we also abbreviated

$$\begin{aligned} \mathcal{K}_i &= \int_{CY_3} \omega_i \wedge J \wedge J = \mathcal{K}_{ijk} \nu^j \nu^k , \\ \mathcal{K}_{ij} &= \int_{CY_3} \omega_i \wedge \omega_j \wedge J = \mathcal{K}_{ijk} \nu^k . \end{aligned} \tag{2.8}$$

Finally let us discuss the kinetic terms of the scalar fields in the action (2.4). Let us denote by φ^α the $(h^{(1,1)} - 1)$ scalar fields which parameterize the hypersurface $\mathcal{K} = 1$. The metric $g^{(5)}_{\alpha\beta}$ which appears in (2.4) is therefore the induced metric on that hypersurface, which is given by [22, 24]

$$g^{(5)}_{\alpha\beta} \ = \ g_{ij} \, \frac{\partial \nu^i}{\partial \varphi^\alpha} \frac{\partial \nu^j}{\partial \varphi^\beta} |_{\mathcal{K}=1} , \qquad \alpha, \beta = 1, \ldots, h^{(1,1)} - 1 . \tag{2.9}$$

For the purpose of our paper it is of interest to also discuss possible (global) isometries of the moduli space of the scalars in the vector multiplets. Following [22] let us consider the infinitesimal linear transformations

$$\nu^i \to \nu^i - \epsilon M^i_j \nu^j , \tag{2.10}$$

where the M^i_j are constant and elements of a Lie Algebra. Since the space of scalar fields in vector multiplets is defined on the hypersurface $\mathcal{K} = 1$ the transformation (2.10) is constrained by the requirement

$$\delta \mathcal{K} = 0 . \tag{2.11}$$

Inserting (2.10) into (2.7) one arrives at [22]

$$M^l_i \mathcal{K}_{jkl} + M^l_j \mathcal{K}_{kil} + M^l_k \mathcal{K}_{ijl} = 0 , \tag{2.12}$$

[5] The same metric g_{ij} will also appear in the four-dimensional effective action which we discuss in the next section. In this case it is the metric on a complex special Kähler manifold, since in $d = 4$ the scalar fields in the vector multiplets are complex and furthermore they necessarily span a special Kähler manifold.

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which states that $\mathcal{K}_{ijk}$ is an invariant tensor of the Lie Algebra. Inserting (2.10) into (2.8) and (2.6) one also computes

$$\delta\mathcal{K}_{ij} = \epsilon M_i^k \mathcal{K}_{kj} + \epsilon M_j^k \mathcal{K}_{ik} \ , \qquad \delta\mathcal{K}_i = \epsilon M_i^j \mathcal{K}_j \ , \tag{2.13}$$

and

$$\delta g_{ij} = \epsilon M_i^k g_{kj} + \epsilon M_j^k g_{ik} \ . \tag{2.14}$$

By assigning the transformation law (2.10) also to the A^i one immediately sees the invariance of the last two terms in the action (2.4). The invariance of the second term in (2.4) is less obvious but has been established in [22]. A quick intuitive argument goes as follows. The full kinetic term on the Calabi-Yau moduli space of Kähler deformations $g_{ij}\partial_\mu \nu^i \partial^\mu \nu^j$ is clearly invariant under (2.10) and (2.14). Since the second term in (2.4) differs from the one above by a kinetic term for the volume modulus $\partial_\mu \mathcal{K} \partial^\mu \mathcal{K}$, which is trivially invariant due to (2.11), it follows that the the kinetic term for the Kähler moduli parameterizing the hypersurface $\mathcal{K} = 1$ is also invariant under the transformation (2.10). Therefore the action (2.4) has a global symmetry for any M_j^i which solves the constraint (2.12).

For generic $\mathcal{K}_{ijk}$, eq. (2.12) has no solutions, or in other words, a generic $\mathcal{K}_{ijk}$ is not an invariant tensor of any Lie Algebra. Let us therefore turn to a specific situation where global isometries do arise, which will be used in the next subsections. The case that we will discuss in detail is the special class of $K3$-fibred Calabi-Yau threefolds (over a $\mathbf{P_1}$ base) [6]. If we denote by ν^1 the volume of the base, then for this class of manifolds the intersection numbers obey $\mathcal{K}_{11i} = 0$. Furthermore, if the $\mathbf{P_1}$ is taken large, i.e. $\nu^1 \gg \nu^{i\neq 1}$ for fixed $\mathcal{K}$, then the moduli space is the scalar manifold [22 – 24]

$$M_V = \mathrm{SO}(1,1) \times \frac{\mathrm{SO}(1, h^{(1,1)} - 2)}{\mathrm{SO}(h^{(1,1)} - 2)} \ . \tag{2.15}$$

The isometry group of this space is $\mathrm{SO}(1,1) \times \mathrm{SO}(1, h^{(1,1)} - 2)$ and in section 2.5 we discuss in detail the corresponding solutions of (2.12). A discrete subgroup of this isometry group, $\mathrm{SO}(1, h^{(1,1)} - 2, \mathbf{Z})$, is known as the U-duality group which is an exact symmetry of these compactifications.

2.2 Seven dimensional manifolds with SU(3) structure

In the previous section we briefly reviewed Calabi-Yau compactifications of M-theory. Let us now turn to compactifications on seven-dimensional manifolds X_7 with SU(3) structure. They can be characterized by a triplet of globally defined and SU(3)-invariant tensors $\{V, J, \Omega\}$, where V is a one-form, J is a two-form and Ω is a three-form [2, 26]. This triplet is constrained to satisfy the compatibility relations

$$\begin{aligned} J \wedge J \wedge J &= \frac{3i}{4}\, \Omega \wedge \bar{\Omega} \ , \\ \Omega \wedge J = V \lrcorner J \ &= \ V \lrcorner \Omega \ = \ 0 \, , \end{aligned} \tag{2.16}$$

where $\lrcorner$ denotes contraction of indices.

Due to the existence of the one-form V, one can define an almost product structure, in that locally the metric can be split as

$$ds_7^2(y,z) = ds_6^2(y,z) + V^2(y,z)\,, \tag{2.17}$$

where y are the coordinates of the six-dimensional component Y_6 and z is the coordinate of the one-dimensional component. On Y_6 the two-form J defines an almost complex structure (by raising one index with the metric) and it is a $(1,1)$-tensor with respect to it. Similarly Ω is a $(3,0)$ form, and together they define the standard SU(3) structure on a six-dimensional space.

The manifold X_7 can be characterized by the non-vanishing intrinsic torsion classes. They are defined by dV, dJ, and $d\Omega$, and can be decomposed into irreducible SU(3) representations. One finds 13 torsion classes denoted R, $c_{1,2}$, $V_{1,2,3}$, $W_{1,2}$, $A_{1,2}$, T and $S_{1,2}$ in [26], defined by

$$\begin{aligned}
dV &= RJ + \bar{W}_1 \lrcorner \Omega + W_1 \lrcorner \bar{\Omega} + A_1 + V \wedge V_1\,,\\
dJ &= \frac{2i}{3}\left(c_1\Omega - \bar{c}_1\bar{\Omega}\right) + J \wedge V_2 + S_1 + V \wedge \left[\frac{1}{3}\left(c_2 + \bar{c}_2\right) J + \bar{W}_2 \lrcorner \Omega + W_2 \lrcorner \bar{\Omega} + A_2\right],\\
d\Omega &= c_1 J \wedge J + J \wedge T + \Omega \wedge V_3 + V \wedge \left[c_2\Omega - 2J \wedge W_2 + S_2\right]\,.
\end{aligned} \tag{2.18}$$

As we already stated we do not compactify on generic SU(3) structure manifolds with all torsion classes non-zero. Instead we focus on manifolds which can be viewed as Calabi-Yau threefolds CY_3 fibred over a circle S^1. With these specifications our setup is closely related to the case of a six-dimensional torus T^6 fibred over a circle. Such backgrounds were discussed in detail in [18, 19] and in the following we can draw on their results.

We parameterize the S^1 direction by the coordinate $z \in [0,1)$, while the radius of the circle is given by the value of the dilaton e^{ϕ}, where $V = e^{\phi}dz$. We further constrain the fibration such that when going around the S^1 only the second cohomology $H^{(1,1)}(CY_3)$ is twisted by a matrix γ, while the third cohomology $H^3(CY_3)$ is unaffected. In this way we ensure that the hypermultiplet sector, which is governed by $H^3(CY_3)$, coincides with that of a $CY_3 \times S^1$ compactification. On the other hand, as we saw in the previous section, the vector multiplets are determined by $H^{(1,1)}(CY_3)$ and hence they do feel the twisting.

As in the previous section we denote the elements of $H^{(1,1)}$ by ω_i but now they also depend on the circle coordinate z, or in other words we have a set of $\omega_i(y,z)$. However, the structure of the fibration is not arbitrary but constrained by a consistency condition. If we choose a specific basis at (say) $z = 0$, it rotates as we move in the z direction. After a full circle, the ω_i must come back to an equivalent theory, i.e., the 5 dimensional theory returns to itself up to a discrete U-duality transformation [18].[6] We already briefly discussed the U-duality group $\Gamma(\mathbf{Z})$ of M-theory compactified on CY_3 at the end of the last subsection and here it appears as the group of monodromies as we go around the circle[7]

$$\omega_i \to \gamma_i^{\,j}\,\omega_j\,, \qquad \gamma_j^{\,i} \in \Gamma(\mathbf{Z})\,. \tag{2.19}$$

[6] By U-duality we broadly refer to the group of discrete gauge transformations of the theory. We implicitly assume that all discrete global symmetries are actually gauged [27].

[7] In the last section we noted that for compactifications which have a heterotic dual the U-duality group is $\Gamma(\mathbf{Z}) = \mathrm{SO}(1, h^{(1,1)} - 2, \mathbf{Z})$, but the analysis of this section holds for arbitrary $\Gamma(\mathbf{Z})$.

In principle, only this global information exists. However, it is convenient to choose an infinitesimal form of this relation by twisting the basis ω_i by a constant matrix, M^i_j, in the continuous group $\Gamma(\mathbf{R})$ as we go along the circle. In this case we write

$$\gamma = e^M, \qquad M \in \Gamma(\mathbf{R}) \ , \tag{2.20}$$

and the infinitesimal version of (2.19) becomes

$$\omega_i(y, z + \epsilon) = \omega_i(y, z) + \epsilon M_i^j \omega_j(y, z) \ . \tag{2.21}$$

Since on the Calabi-Yau slice the ω_i continue to be harmonic (and therefore closed), (2.21) can also be expressed by the differential relation

$$d\omega_i = M_i^j \, \omega_j \wedge dz \ . \tag{2.22}$$

Equations (2.21) and (2.22) hold whenever the monodromy is evenly distributed along the S^1. This turns out to be useful for carrying out a KK reduction even when there is no continuous isometry. In this case (2.21) will in general not be a solution of the equations of motion, but it is still a useful ansatz for analyzing the compactification. A specific example where this ansatz gives a solution arises when we consider degenerations of the Calabi-Yau compactification in which a continuous version of the U-duality appears as an approximate global symmetry $\Gamma(\mathbf{R})$. As we discussed at the end of section 2.1 this situation occurs, for example, when the base in the $K3$-fibred CY_3 is large. In this case the matrix M satisfies (2.12), and (2.21) expresses a translation invariance along the S^1.

However, generically the full theory does not have the continuous symmetry and the only real information is the global data in the monodromy γ. The approach that we will take is first to discuss the situation with a continuous symmetry, obtain a quantitative understanding of what this process of twisting does, and afterwards indicate (in section 2.6) why going away from this limit does not change the qualitative picture.

Let us define the Calabi-Yau intersection numbers exactly as in (2.5), but now with z-dependent $\omega_i(y, z)$. In this case the $\mathcal{K}_{ijk}$ can a priori also be z-dependent. However, inserting (2.21) into (2.5), we see that precisely when there is an isometry the z-dependence cancels out due to (2.12). Note that the fact that the same matrix M_i^j appears in (2.12) and (2.21) establishes the connection between an isometry in the space-time effective theory and the translational symmetry of the fibration of the $(1,1)$-forms along the S^1-circle.

In the Kaluza-Klein reduction which we perform in the next section we encounter the seven-dimensional integral

$$\hat{\mathcal{K}}_{ijk} = \int_{X_7} \omega_i \wedge \omega_j \wedge \omega_k \wedge dz \ , \tag{2.23}$$

which are the intersection numbers defined on the entire X_7. They coincide with the $\mathcal{K}_{ijk}$ precisely when (2.12) holds. In this case the $\mathcal{K}_{ijk}$ are z-independent, and thus the integral in (2.23) trivially factorizes. Note that the condition (2.12) also arises in this case from the requirement of global consistency of (2.22)

$$\int_{X_7} d(\omega_i \wedge \omega_j \wedge \omega_k) = 0 \ . \tag{2.24}$$

It is also useful to note that all the other Calabi-Yau moduli space quantities defined in equations (2.6), (2.7) and (2.8) have, due to (2.12), similar definitions in terms of seven-dimensional integrals. In particular the Calabi-Yau volume can also be defined as

$$\mathcal{K} = \frac{1}{6}\int_{X_7} J \wedge J \wedge J \wedge dz \ . \tag{2.25}$$

The volume of the full seven-dimensional manifold X_7 differs from this one by a dilaton factor, which, when the dilaton is independent of the X_7 coordinates is equal to

$$\hat{\mathcal{K}} = \frac{1}{6}\int_{X_7} J \wedge J \wedge J \wedge V = e^{\phi}\mathcal{K} \ . \tag{2.26}$$

In analogy with (2.3) we expand J according to

$$J \ = \ v^i \, \omega_i(y, z) \, , \tag{2.27}$$

where the v^i are again constant but now there is a z-dependence in ω_i. The v^i will appear as scalar fields in the four-dimensional effective action. Note that J is not invariant under translation in the z-direction, but it comes back to itself when we go all the way around the circle. This follows from the fact that we identify the manifold under $z \to z + 1$ together with (2.19). As a consequence J is globally defined on X_7.

As we will see in the next subsection, it is the z-dependence of the ω_i in (2.27) which generates mass terms for the fields v^i in the four-dimensional effective action. Let us note that this can also be seen from a Scherk-Schwarz point of view [28] where one first compactifies to five dimensions on Calabi-Yau manifolds as in the previous section and then, in the subsequent compactification to four dimensions, gives the five-dimensional scalar fields ν^i a monodromy as one moves around the circle such that their z-dependence is given by $\nu^i(z + \epsilon) = \nu^i(z) + \epsilon M^i_j \, \nu^j(z)$. Thus the relation between ν^i and v^i is simply $\nu^i(z) = (e^{zM})^i_j \, v^j$.

Inserting eq. (2.27) into eq. (2.25) and using eq. (2.23) and $\hat{\mathcal{K}}_{ijk} = \mathcal{K}_{ijk}$ we obtain $\mathcal{K} = \frac{1}{6}\mathcal{K}_{ijk}v^i v^j v^k$ exactly as in (2.7), but now in terms of the parameters v^i instead of ν^i. Similarly, the metric on the moduli space of Kähler deformations can be defined as

$$g_{ij} = \frac{1}{4\mathcal{K}}\int_{X_7} \omega_i \wedge *\omega_j \ , \tag{2.28}$$

with no dilaton prefactor, which is in agreement with the metric Ansatz (2.31) we shall consider in the next section. One can show that it coincides with the metric given in eq. (2.6) with the replacement $\nu^i \to v^i$ in (2.7) and (2.8).

Before we turn to the details of the KK-reduction let us determine the non-trivial torsion classes in (2.18) for the fibration characterized by eq. (2.21) or equivalently by eq. (2.22). Using the expansion (2.27) with the forms ω_i satisfying (2.22) we find

$$dJ = v^i M_i^j \, \omega_j \wedge dz \, , \tag{2.29}$$

which shows that the M_i^j parameterize the non-vanishing intrinsic torsion. Comparison with (2.18) reveals that the only torsion classes which can be non-trivial are A_2 and $\mathrm{Re}\, c_2$.

Actually, for the case at hand, it can be shown that Re c_2 vanishes and the only torsion class which is present is A_2. This can be seen by writing (2.29) in components and contracting with J^{mn}. Using the SU(3) structure consistency relation $V_m J^{mn} = 0$, (2.16), and the fact that A_2 in (2.29) is primitive, i.e. $(A_2)_{mn} J^{mn} = 0$, leaves us with the following relation for Re c_2

$$\mathrm{Re}\; c_2 \sim v^i M_i^j (\omega_j)_{mn} J^{mn} \; . \tag{2.30}$$

For Calabi-Yau manifolds, the contraction of the (1,1) forms ω_j with J was computed in [25] and shown to be proportional to $\mathcal{K}_{jkl} v^k v^l$. Inserting this into the above equation, the vanishing of the torsion class Re c_2 is simply a consequence of the constraint (2.12). Note that for $M = 0$ the two-form J is closed, the fibration is trivial and X_7 is the product manifold $CY_3 \times S^1$.

2.3 Kaluza-Klein reduction of M-theory on X_7

We can now proceed with one of the main parts of this paper, namely the compactification of M-theory, or rather eleven-dimensional supergravity, on seven-dimensional manifolds with SU(3) structure. As explained before, we concentrate on the vector multiplet sector and ignore the hypermultiplets in our analysis.

The starting point is the eleven-dimensional action (2.1). Since on seven-dimensional manifolds with SU(3) structure we can define an almost product structure we consider the following Ansatz for the metric

$$G_{MN} = \begin{pmatrix} e^{4\phi/3}\left(\frac{1}{\mathcal{K}} G_{\mu\nu} + A^0_\mu A^0_\nu\right) & 0 & -e^{4\phi/3} A^0_\mu \\ 0 & e^{-2\phi/3} G_{mn} & 0 \\ -e^{4\phi/3} A^0_\nu & 0 & e^{4\phi/3} \end{pmatrix} , \tag{2.31}$$

where $G_{\mu\nu}$ denotes the 4d metric, G_{mn} is the metric on the Calabi-Yau manifold, A^0_μ is the $4d$ graviphoton and ϕ the dilaton.[8] The scalar fields arising from the Calabi-Yau metric correspond to the deformations of J which we denoted by v^i in (2.27), as well as the deformations of Ω. The dilaton factors are chosen in such a way that we end up in the four dimensional Einstein frame. The factor $1/\mathcal{K}$ – with $\mathcal{K}$ defined in (2.25) — in front of the four-dimensional metric has been introduced to account for the additional Calabi-Yau volume factor which appears in front of the Einstein-Hilbert term after performing the integral over the internal manifold.

Next we expand the three-form potential according to

$$\hat{C}_3 = \tilde{C}_3 + B \wedge dz + \tilde{A}^i \wedge \omega_i + b^i \omega_i \wedge dz + \cdots \; , \tag{2.32}$$

where $\tilde{C}_3$ is a three-form in four dimensions, B is a two-form, $\tilde{A}^i$ are vector fields and b^i are scalars. The ... stand for additional scalar fields that arise when $\hat{C}_3$ is expanded in a basis of $H^3(CY_3)$, which, together with the complex structure deformations, the dual of B, and the dilaton ϕ, fill out $h^{(1,2)} + 1$ hypermultiplets, and we omit them from our

[8] The above Ansatz includes only zero modes, and therefore we omitted the off-diagonal components which involve one-forms on CY_3, since they lead to massive excitations.

further discussion.[9] We do keep the gravity multiplet which includes the graviton and the graviphoton A^0, and the $h^{(1,1)}$ vector multiplets which include the vector fields $\tilde{A}^i$ and the complex scalars $x^i = b^i + iv^i$. Note that compared to the five-dimensional case discussed in section 2.1, there is an additional vector multiplet and the Kähler moduli are complexified. Thus, in the four-dimensional effective action all Kähler moduli, including the Calabi-Yau volume, are in vector multiplets.

In the compactification process it is useful to keep track of the isometries of the internal manifold X_7 since they become gauge transformations in the effective theory. Let us first recall the situation for compactifications on $CY_3 \times S^1$. In this case there is an isometry corresponding to constant shifts $z \to z + \epsilon$ of the S^1 coordinate. Promoting the parameter to be space-time dependent $\epsilon \to \epsilon(x^\mu)$, the compactification Ansatz given in eqs. (2.31) and (2.32) changes. Keeping $\hat{C}_3$ and the ten-dimensional line-element ds^2_{10} invariant induces the local gauge transformations

$$A^0 \to A^0 + d\epsilon \ , \qquad \tilde{C}_3 \to \tilde{C}_3 - B \wedge d\epsilon \ , \qquad \tilde{A}^i \to \tilde{A}^i - b^i d\epsilon \ . \tag{2.33}$$

However, the fact that the fields $\tilde{C}_3$ and $\tilde{A}^i$ transform is an artefact of the expansion (2.32) and one can define the gauge-invariant fields

$$C_3 = \tilde{C}_3 + B \wedge A^0 \ , \qquad A^i = \tilde{A}^i + b^i A^0 \ . \tag{2.34}$$

In the case of a non-trivial fibration of the Calabi-Yau over the circle, as considered in eq. (2.21), these fields are no longer invariant. However, the main property of eq. (2.34) is that the transformations of C_3 and A^i do not contain the derivative of the transformation parameter ϵ, and therefore we will keep the same definitions in the following. Using eqs. (2.21) and (2.32) we can easily see that the fields A^i and b^i arising from the expansion of $\hat{C}_3$ acquire a non-trivial gauge transformation, but the transformation law of the graviphoton is unchanged.

Exactly as for $\hat{C}_3$, we also need to keep J, defined in eq. (2.27), gauge-invariant. Since the basis of $(1,1)$ forms ω_i changes according to eq. (2.21), we need to assign a transformation law similar to (2.10) also to the fields v^i. Another way of saying this is that our background is not invariant under arbitrary shifts of z, but, as shifts of z are gauge symmetries, we must assign a transformation law to the v^i. Altogether we thus have

$$\begin{aligned} A^0 &\to A^0 + d\epsilon \ , \qquad & A^i &\to A^i - \epsilon M^i_j A^j \ , \\ v^i &\to v^i - \epsilon M^i_j v^j \ , & b^i &\to b^i - \epsilon M^i_j b^j \ . \end{aligned} \tag{2.35}$$

Note that unlike the $N = 2$ gauged supergravities encountered so far in string compactifications, the symmetry (2.35) is not necessarily a Peccei-Quinn shift symmetry which is usually gauged in these cases. Moreover, this gauge symmetry is generically spontaneously broken due to the non-vanishing vacuum expectation values of the Kähler moduli v^i.

[9] The couplings of the hypermultiplets in the $N = 2$ low energy effective action can be found, for example, in [29, 13].

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In addition, the four-dimensional effective theory sees the remnant of the three-form gauge invariance $\hat{C}_3 \to \hat{C}_3 + d\Lambda_2$ which is manifest in the action (2.1). Choosing $\Lambda_2 = \eta^i \omega_i$ and using (2.22) we obtain the following transformation laws

$$\begin{aligned} A^0 &\to A^0 \,, \quad A^i \to A^i + d\eta^i + M^i_j \eta^j A^0 \,, \\ v^i &\to v^i \,, \qquad b^i \to b^i + M^i_j \eta^j \,. \end{aligned} \tag{2.36}$$

The parameters (ϵ, η^i) together form $h^{(1,1)} + 1$ local gauge parameters. From the transformations displayed in (2.35) and (2.36) we already see the non-Abelian character of the gauge transformations when $M \neq 0$.

To derive the four-dimensional action we insert eqs. (2.31) and (2.32) into (2.1) and perform the integrals over the internal manifold. Let us first concentrate on the last two terms in the action (2.1), and postpone the compactification of the Ricci scalar to the end of this section.

To make our task easier let us first compute the field strength $\hat{F}_4$ by taking the exterior derivative of eq. (2.32). Using (2.22) and the definitions (2.34) we find

$$\begin{aligned} \hat{F}_4 = d\hat{C}_3 = &(dC_3 - B \wedge F^0) + H \wedge (dz - A^0) \\ &+ (F^i - b^i F^0) \wedge \omega_i + Db^i \wedge \omega_i \wedge (dz - A^0) + \cdots \,, \end{aligned} \tag{2.37}$$

where we defined

$$F^0 = dA^0 \,, \qquad F^i = dA^i - M^i_j \, A^j \wedge A^0 \,, \qquad Db^i = db^i - M^i_j \, (A^j - b^j A^0) \,. \tag{2.38}$$

The reason we have formally performed the expansion in the forms $dz - A^0$ is that in this basis the metric (2.31) is block diagonal, and therefore in computing $(\hat{F}_4)^2$ only the square of the individual terms in (2.37) will appear and no mixed terms will be present. Note that in four dimensions C_3 is not a dynamical field and therefore we will discard its contribution in the following. In general, a proper dualization should be performed, but this has implications only on the hypermultiplet sector and is therefore not of interest for us. With these things in mind we obtain

$$\int_{X_7} \hat{F}_4 \wedge * \hat{F}_4 = e^{-4\varphi} H_3 \wedge * H_3 + 4\mathcal{K} g_{ij} (F^i - b^i F^0) \wedge * (F^j - b^j F^0) + 4 g_{ij} Db^i \wedge * Db^j + \cdots \,, \tag{2.39}$$

where the metric g_{ij} was defined in eq. (2.28) and φ denotes the four dimensional dilaton defined as $e^{-2\varphi} = e^{-2\phi}\mathcal{K}$. For the Chern-Simons term in (2.1) one finds after a straightforward but somewhat lengthy calculation

$$\begin{aligned} \int_{X_7} \hat{C}_3 \wedge \hat{F}_4 \wedge \hat{F}_4 \; = &\, 3F^i \wedge F^j b^k \mathcal{K}_{ijk} - 3F^i \wedge F^0 b^j b^k \mathcal{K}_{ijk} \\ &+ F^0 \wedge F^0 b^i b^j b^k \mathcal{K}_{ijk} + 2M^k_i A^i \wedge A^l \wedge F^j \mathcal{K}_{jkl} \,, \end{aligned} \tag{2.40}$$

where $\mathcal{K}_{ijk}$ are the Calabi-Yau intersection numbers defined in (2.5) which can also be obtained from (2.23).

Let us check explicitly that the individual terms in eq. (2.39) are invariant under the gauge transformations (2.35) and (2.36). Under (2.35) the quantities defined in eq. (2.38) transform as

$$\delta Db^i = -\epsilon M^i_j Db^j \ , \qquad \delta F^i = -\epsilon M^i_j F^j \ , \qquad \delta F^0 = 0 \ . \tag{2.41}$$

Together with the transformation (2.14) of the moduli space metric, this shows that the terms in eq. (2.39) are (individually) invariant. Under the gauge transformation (2.36), the covariant derivatives Db^i are invariant as can be checked from (2.38). The field strengths F^i, F^0 are not individually invariant, but the combinations

$$\check{F}^i = F^i - b^i F^0 \ , \tag{2.42}$$

which appear in (2.39), are invariant. This completes the proof of the gauge invariance of the expression (2.39).

We can similarly check the gauge invariance of (2.40). For the transformation (2.35) it follows straightforwardly from eq. (2.41) and the constraint (2.12) that each term in (2.40) is invariant individually. To check the invariance under the transformation (2.36) is also straightforward but a bit more tedious. The important difference to note is that the gauge invariance (2.36) only holds for the sum of all the terms in eq. (2.40) but not for the individual terms. We come back to this issue in section 2.4.

The next step is to compactify the Ricci scalar in the action (2.1). For $CY_3 \times S^1$ the answer is well known [23] and yields the kinetic terms for the moduli v^i, a contribution to the kinetic terms of the graviphoton A^0 and the kinetic term for the dilaton. For the case of a non-trivial fibration the moduli are charged under the isometry of the circle and the corresponding gauge transformation is given in (2.35). This in turn leads to a coupling of the moduli to the graviphoton and a scalar potential. The generic formulae for this case are worked out in [28] and we can borrow some of their results. One finds

$$\frac{1}{2}\int_{X_7} \hat{R} * 1 = \frac{1}{2} R_4 * \mathbf{1} - g_{ij} Dv^i \wedge * Dv^j - \mathcal{K} F^0 \wedge * F^0 - d\varphi \wedge * d\varphi - V \ . \tag{2.43}$$

This is a straightforward generalization of the result obtained in $CY_3 \times S^1$ compactifications, in that the derivatives for the charged moduli are replaced by covariant derivatives

$$Dv^i = dv^i + v^j M^i_j A^0 \ . \tag{2.44}$$

The derivation of the scalar potential V is less obvious and an explicit calculation of the internal Ricci scalar is necessary. Note that this gives in fact the only contribution to the potential as eqs. (2.39) and (2.40) contain no terms without four-dimensional derivatives. Let us therefore compute the scalar curvature for the internal part of the metric which can be read off from (2.31)

$$G_{\text{int}} = e^{-2\phi/3} \begin{pmatrix} G_{mn}(y,z) & 0 \\ 0 & e^{2\phi} \end{pmatrix} \ . \tag{2.45}$$

From the seven-dimensional point of view, the overall dilaton factor is irrelevant as this is just a constant, but it will be important for the normalization of the potential. Using the fact that the Ricci tensor of the Calabi-Yau slices vanishes we find

$$R_7 = -e^{-4\phi/3}\left[\partial_z(G^{mn}\partial_z G_{mn}) + \frac{1}{4}(G^{mn}\partial_z G_{mn})^2 + \frac{1}{4}G^{mn}G^{pq}\partial_z G_{mp}\partial_z G_{nq}\right] . \tag{2.46}$$

In order to proceed we split the metric into a background piece G^0_{mn}, which is constant in z, and the moduli dependent part ΔG_{mn} which does depend on z:

$$G_{mn} = G^0_{mn} + \Delta G_{mn} . \tag{2.47}$$

As explained before, the fibration structure we consider is such that the complex structure deformation sector is not influenced by the additional z direction, and we are only interested in the dependence on the Kähler moduli v^i. In complex coordinates they arise from the $(1,1)$ components of the metric via

$$\Delta G_{a\bar{b}} = -iv^i(\omega_i)_{a\bar{b}} , \qquad a, \bar{b} = 1, 2, 3 . \tag{2.48}$$

Using eq. (2.21) we immediately find

$$\partial_z \Delta G_{a\bar{b}} = -iv^i M_i^j(\omega_j)_{a\bar{b}} . \tag{2.49}$$

From the fact that ω_j is a harmonic $(1,1)$-form on the Calabi-Yau threefold, one shows, following ref. [25], that $G^{a\bar{b}}(\omega_j)_{a\bar{b}} = \frac{i}{2}\mathcal{K}_j/\mathcal{K}$, where eqs. (2.7) and (2.8) were used. Combining this with eq. (2.49) gives

$$G^{mn}\partial_z G_{mn} = \mathcal{K}_{jkl} M_i^j v^i v^k v^l = 0 , \tag{2.50}$$

as a consequence of the constraint (2.12). Therefore the only contribution to the four-dimensional potential comes from the last term in (2.46). Inserting eq. (2.49) into eq. (2.46) we arrive at

$$\frac{1}{2}\int_{X_7} R_7 = -\frac{1}{4}e^{-4\phi/3} M_i^k M_j^l v^i v^j \int_{X_7} \omega_k \wedge *\omega_l . \tag{2.51}$$

Using eqs. (2.28) and (2.31), and taking into account the rescaling of the four-dimensional metric, we finally obtain the potential (in the Einstein frame)

$$V = \frac{1}{\mathcal{K}}\, v^i v^j M_i^k M_j^l g_{kl} . \tag{2.52}$$

2.4 Consistency with $N = 2$ supergravity

In order to check the consistency with $N = 2$ supergravity (reviewed in appendix A) we have to write the resulting four-dimensional action in the general form (A.16). Putting together eqs. (2.39), (2.40) and (2.43) we obtain the action in four dimensions for the bosonic fields in the gravity and vector- multiplets

$$\begin{aligned} S = \int_{M_4} \Big[& \frac{1}{2}\, R * 1 - g_{ij} Dx^i \wedge * D\bar{x}^j - V \\ & + \frac{1}{4}\,\mathrm{Im}\mathcal{N}_{IJ} F^I \wedge * F^J + \frac{1}{4}\,\mathrm{Re}\mathcal{N}_{IJ} F^I \wedge F^J - \frac{1}{6}\, M_i^l \mathcal{K}_{jkl} A^i \wedge A^j \wedge dA^k \Big] , \end{aligned} \tag{2.53}$$

where the metric g_{ij} was defined in (2.28). It is a special Kähler metric derived from the Kähler potential given in (A.1) for the prepotential

$$\mathcal{F}(X) = -\frac{1}{6}\frac{\mathcal{K}_{ijk}X^iX^jX^k}{X^0} . \tag{2.54}$$

The X^I, $I = 0,\dots,h^{(1,1)}$, are projective coordinates which are related to the scalar fields via so-called special coordinates x^i given by

$$x^i = \frac{X^i}{X^0} = b^i + iv^i , \qquad i = 1,\dots h^{(1,1)} . \tag{2.55}$$

The prepotential (2.54) also determines the gauge coupling matrix $\mathcal{N}$ via (A.3), and one finds

$$\begin{aligned} \mathrm{Re}\,\mathcal{N}_{00} &= -\frac{1}{3}\mathcal{K}_{ijk}b^ib^jb^k , & \mathrm{Re}\,\mathcal{N}_{i0} &= \frac{1}{2}\mathcal{K}_{ijk}b^jb^k , & \mathrm{Re}\,\mathcal{N}_{ij} &= -\mathcal{K}_{ijk}b^k , \\ \mathrm{Im}\,\mathcal{N}_{00} &= -\mathcal{K}(1+4g_{ij}b^ib^j) , & \mathrm{Im}\,\mathcal{N}_{i0} &= 4\mathcal{K}g_{ij}b^j , & \mathrm{Im}\,\mathcal{N}_{ij} &= -4\mathcal{K}g_{ij} . \end{aligned} \tag{2.56}$$

The field strengths in eq. (2.53) are given by

$$F^I = dA^I + \frac{1}{2}f^I_{JK}A^J\wedge A^K , \qquad \text{with} \qquad f^0_{IJ} = 0 = f^k_{ij} , \qquad f^j_{i0} = -M^j_i , \tag{2.57}$$

while the covariant derivatives read

$$Dx^i = dx^i - k^i_I A^I , \qquad \text{with} \qquad k^j_0 = -x^k M^j_k , \qquad k^j_i = M^j_i . \tag{2.58}$$

These holomorphic Killing vectors can be obtained via (A.11) from the Killing prepotentials

$$P_0 = -x^i M^j_i K_j , \qquad P_i = M^j_i K_j , \tag{2.59}$$

where $K_j = \partial_j K$ is the first derivative of the Kähler potential. The consistency of the non-Abelian gauge algebra can be checked in that eq. (A.14) is fulfilled and we have

$$[k_i, k_j] = 0 = [k_0, k_0] , \qquad [k_i, k_0] = -M^j_i k_j , \tag{2.60}$$

corresponding to a semi-direct sum of two Abelian sub-algebras.[10] Finally, using (2.58) it is easy to see that the potential (2.52) is consistent with (A.15).

Except for the last term in eq. (2.53) everything looks like a standard $N = 2$ gauged supergravity as spelled out in ref. [31]. The last term is also known, and has to be introduced in the action (in order to make it gauge-invariant) whenever the prepotential is not invariant under the gauge transformations, but transforms into a second order polynomial in X with real coefficients [32]. Inserting the transformation (2.36) into the definition of the projective coordinates (2.55) we find that the prepotential (2.54) changes as

$$\delta_\eta\mathcal{F} = -\frac{1}{2}\eta^i M^l_i \mathcal{K}_{ljk}X^jX^k , \tag{2.61}$$

[10] We thank the referee of this paper for pointing out that (2.60) is also a solvable Lie algebra (for a definition see, for example, [30]).

which is precisely of the form (A.17) with $C_{ijk} = \frac{1}{2} M_i^l \mathcal{K}_{ljk}$. Note that for the specific structure constants given in eq. (2.57), the last term of eq. (A.18) vanishes, which explains why such a term is absent from eq. (2.53).

Before we continue it is worthwhile to stress that the vector multiplet geometry on the M-theory side specified by the prepotential (2.54) is exact, since the "dilaton" (the radius of the M-theory circle) is part of a hypermultiplet, and therefore cannot correct this geometry. The same holds for the gauging as specified in (2.58).

2.5 $K3$-fibred Calabi-Yau threefolds

So far our discussion was generic, in that we did not specify the intersection numbers $\mathcal{K}_{ijk}$ and the matrix M_i^j. We did however assume that the seven-dimensional X_7 is a fibred product of a Calabi-Yau threefold CY_3 over a circle, and that the CY_3 is such that a continuous isometry of the form (2.21) exists. In this section we discuss more concretely the specific case of $K3$-fibred Calabi-Yau threefolds; type IIA string theory compactified on such threefolds is dual to heterotic string theory compactified on $K3 \times T^2$.

$K3$-fibred Calabi-Yau threefolds consist of $K3$ fibres over a $\mathbf{P_1}$ base [6]. The volume of the base in string units is identified with the dilaton on the heterotic side. Furthermore, two additional two-cycles in the $K3$, related to the heterotic torus, can be singled out. Let us denote these three special cycles by 1, 2 and 3, while the rest of the two-cycles are denoted by an index a. In the limit of a large $\mathbf{P_1}$ base (i.e. large heterotic dilaton) the prepotential (2.54) becomes

$$\mathcal{F} = \frac{X^1(X^2X^3 - X^aX^a)}{X^0} \tag{2.62}$$

and so the only non-vanishing intersection numbers for the Calabi-Yau threefold are [6]

$$\mathcal{K}_{123} = -1 \,, \qquad \mathcal{K}_{1ab} = 2\delta_{ab} \,, \qquad a, b = 4, \ldots, h^{(1,1)} \,. \tag{2.63}$$

Inserting eq. (2.62) into eq. (A.1) and computing the corresponding Kähler metric one sees that this factorizes and becomes the metric on the space

$$M_V = \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(2, h^{(1,1)} - 1)}{\mathrm{SO}(2) \times \mathrm{SO}(h^{(1,1)} - 1)} \,. \tag{2.64}$$

The first factor is spanned by the coordinate x^1 which parameterizes the volume of the $\mathbf{P_1}$ base, while x^2, x^3 and x^a span the second factor. We immediately see that M_V has the continuous isometry group $\mathrm{SU}(1,1) \times \mathrm{SO}(2, h^{(1,1)} - 1)$. As discussed above, in the same limit the five dimensional vector multiplet moduli space has the continuous isometry group $\mathrm{SO}(1,1) \times \mathrm{SO}(1, h^{(1,1)} - 2)$.

As a consequence, we expect that the constraint (2.12) has non-trivial solutions. Indeed, solving eq. (2.12) for the torsion parameters M_i^j, given the intersection numbers (2.63), we find that one can choose to express all matrix elements in terms of $\frac{1}{2}(h^{(1,1)} - 1)(h^{(1,1)} - 2) + 1$ independent parameters

$$m_2 \equiv M_2^2 \,, \qquad m_a \equiv M_a^2 \,, \qquad m_3 \equiv M_3^3 \,, \qquad \tilde{m}_a \equiv M_a^3 \,, \qquad m_b^a \equiv -M_a^b \,, \tag{2.65}$$

where $m^a_b = -m^b_a$. The other matrix elements are then given by

$$\begin{aligned} &M^a_2 = \frac{1}{2}\tilde{m}_a\,, \qquad M^a_3 = \frac{1}{2}m_a\,, \qquad M^a_a = -\frac{1}{2}M^1_1 \;=\; \frac{1}{2}(m_2+m_3)\,, \\ &M^{2,3}_1 = M^a_1 = M^1_a = M^1_{2,3} = M^3_2 = M^2_3 = 0\,. \end{aligned} \tag{2.66}$$

Note that these solutions describe the mixing of $\mathrm{SO}(1,1)\times\mathrm{SO}(1,h^{(1,1)}-2)$ into the gauge symmetry, i.e, we have accounted for the most general monodromy allowed on the circle. However, this is not the most general global symmetry of the four dimensional theory, which can be as large as $\mathrm{SU}(1,1)\times\mathrm{SO}(2,h^{(1,1)}-1)$. In section 3 we discuss how the parameters in (2.66) are related to the dual heterotic background. Before we do so let us return to the situation where the $\mathbf{P_1}$-base is not necessarily large.

2.6 Breaking the continuous isometry

So far our analysis assumed that the Calabi-Yau moduli space has a continuous isometry, or in other words that $\mathcal{K}_{ijk}$ are such that eq. (2.12) has a solution. As we saw this is indeed the case for $K3$-fibred Calabi-Yau manifolds in the large $\mathbf{P_1}$ limit where the moduli space has a continuous $\mathrm{SO}(1,h^{(1,1)}-2)$ symmetry. However, this symmetry is broken (for example, by non-zero intersection numbers $\mathcal{K}_{abc}$ or $\mathcal{K}_{23a}$) to a discrete subgroup $\Gamma(\mathbf{Z}) = \mathrm{SO}(1,h^{(1,1)}-2,\mathbf{Z})$ (which is the T-duality group of the heterotic string) for finite $\mathbf{P_1}$ volume. As we discussed before, the only information that we can really specify at finite $\mathbf{P_1}$ volume is an element $\gamma_i^j \in \Gamma(\mathbf{Z})$, which rotates the $(1,1)$-forms ω_i as described in section 2.2. Of course the absence of the continuous isometry also holds for compactifications on $CY_3\times S^1$ without any monodromy; in this case the corresponding continuous symmetry is broken to a discrete subgroup $\Gamma'(\mathbf{Z})$, which is the T-duality group of the dual heterotic string on $K3\times T^2$. Furthermore in four dimensions type IIA world-sheet instantons also contribute to the breaking of the continuous isometry.

Even though the continuous isometry of the Calabi-Yau moduli space is broken, we want to argue that our M-theory backgrounds retain a subgroup of this isometry. The key difference from the $CY_3\times S^1$ background is that the non-trivial monodromy has the effect that in the four-dimensional effective action part of the would-be isometry (which is indeed an isometry at infinite $\mathbf{P}_1$ volume) is gauged (see eqs. (2.35) and (2.36)). Since it is part of a gauge symmetry, consistency requires that it must persist in the four-dimensional effective action for any value of the parameters — and in particular for finite $\mathbf{P_1}$ volume. To reiterate, this must be true even when the continuous symmetry is not present for the theory without the monodromy, or in the five dimensional effective action.

In order to see in slightly more details how this happens let us first reconsider the computation of the four-dimensional effective action performed in section 2.3. Without the isometry in the Calabi-Yau moduli space the intersection numbers $\mathcal{K}_{ijk}$ defined in eq. (2.5) with ω_i obeying eq. (2.21) are z-dependent and thus vary along the circle. Instead, it is the intersection numbers $\hat{\mathcal{K}}_{ijk}$ defined in (2.23) that appear in the four-dimensional effective action. Now they no longer coincide with the $\mathcal{K}_{ijk}$ as was the case in the presence of a Calabi-Yau isometry. Nevertheless, if we still require that the monodromy is evenly distributed along the circle, or in other words if we continue to impose (2.22) for constant M_i^j, then eq. (2.24) implies

$$M_i^l \hat{\mathcal{K}}_{jkl} + M_j^l \hat{\mathcal{K}}_{kil} + M_k^l \hat{\mathcal{K}}_{ijl} = 0 \ . \tag{2.67}$$

Thus, the Ansatz (2.21) with constant M_i^j implies the presence of an isometry in the moduli space of X_7 even though the isometry of the fibred Calabi-Yau manifold is broken. The existence of this isometry can be viewed as a direct consequence of the gauge symmetry.

The KK-reduction of section 2.3 can be repeated, but now $\hat{\mathcal{K}}_{ijk}$ and the metric defined by (2.28) appear. This metric coincides with the Calabi-Yau moduli space metric (2.6) only for infinite $\mathbf{P_1}$, but differs for finite volume. Therefore the resulting four-dimensional effective action receives small corrections at finite volume. However, these corrections cannot lead to any qualitative changes, since already at large $\mathbf{P_1}$ volume all fields relevant for the gauging are massive, and the corrections just shift their precise mass spectrum.

It would be worthwhile to compute the low energy effective action more explicitly and check its consistency with $N = 2$ supergravity. Furthermore, arguments along the lines of refs. [33, 34] should exist in order to argue that the gauged symmetry is also protected against the breaking coming from the world-sheet instantons. We hope to return to these issues elsewhere.

3. Heterotic string theory compactified on $K3 \times T^2$ with T^2 fluxes

In this section we discuss the heterotic string compactified on $K3 \times T^2$ with the gauge fields having non-trivial flux on the T^2. More specifically we show that the dual background is related to the M-theory compactification discussed in the previous section. We begin by reviewing the heterotic compactification in sections 3.1–3.4, and we present the details of the duality map in section 3.5.

3.1 General properties

Consider heterotic string theory compactified on $K3 \times T^2$. In this subsection we analyze the effect of turning on gauge flux on the T^2 in the low-energy supergravity theory. In particular we want to show that turning on the flux breaks the corresponding gauge symmetry, giving the gauge field a mass proportional to the flux.

In ten dimensions the spectrum of the heterotic string includes a 2-form field B and a gauge field A with field strength F (in either the Spin(32) or the $E_8 \times E_8$ gauge group). The 3-form field strength involves not just the 2-form field, but rather it takes the form:

$$H^{\text{het}} = dB - \frac{\alpha'_{\text{het}}}{4}\omega_3, \tag{3.1}$$

where ω_3 is the Chern-Simons form[11]

$$\omega_3 = \text{tr}\Big(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\Big). \tag{3.2}$$

[11]There is also a gravitational Chern-Simons term in H^{het}, which is of higher order in the Planck constant and will not play a role in our discussion.

The ten-dimensional action includes kinetic terms proportional to

$$\left[-\frac{1}{2}|H^{\text{het}}|^2 - \frac{\alpha'_{\text{het}}}{4}\text{tr}(F^2)\right]. \tag{3.3}$$

Suppose that the compactification to six dimensions on $K3$ breaks the gauge group such that it has a $\mathrm{U}(1)^n$ factor, and consider a background where we turn on a flux for one of the corresponding $\mathrm{U}(1)$ gauge fields A^a on the T^2 ($a = 1, \ldots, n$),

$$\int_{T^2} F^a \equiv f^a \neq 0. \tag{3.4}$$

The six dimensional action includes a term proportional to

$$\int_{R^4\times T^2}\left[\left(dB - \frac{\alpha'_{\text{het}}}{4}A^a \wedge F^a\right)^2 + \frac{\alpha'_{\text{het}}}{2}(F^a)^2\right], \tag{3.5}$$

such that the four dimensional action expanded around the flux background (3.4) includes a term proportional to

$$\int_{R^4}\left[\left(db - \frac{\alpha'_{\text{het}}}{4}f^a A^a\right)^2 + \frac{\alpha'_{\text{het}}V(T^2)^2}{2}(F^a)^2\right], \tag{3.6}$$

where b is the scalar field arising from $\int_{T^2} B$, and $V(T^2)$ is the volume of the T^2. Naively the first term is not gauge-invariant, but in fact the gauge transformation (already in ten dimensions) acts also on the 2-form field, and this transformation in four dimensions takes the form $A^a \to A^a + d\Lambda^a$, $b \to b + \frac{\alpha'_{\text{het}}}{4}f^a\Lambda^a$ such that (3.6) is gauge-invariant.

Both from the form of (3.6) and from the form of the gauge transformation, we see that the $\mathrm{U}(1)$ gauge symmetry is broken, since it acts non-linearly on the scalar field b. The gauge field acquires a mass proportional to f^a, and swallows the scalar field b by the Higgs mechanism. Using (3.6) we see that the mass squared of the gauge field is proportional to $\alpha'_{\text{het}}f_a^2/V(T^2)^2$. In the action we wrote here we set many fields to zero, the full results may be found in [17].

In the previous section we saw that a similar Higgs mechanism in M-theory arises from the non-trivial fibration structure over the M-theory circle. In the following we argue why it is indeed necessary to go to the M-theory description on the dual type IIA side when we add the heterotic fluxes, and afterwards we make the correspondence between the M-theory and the heterotic Higgsing more precise.

3.2 Mapping the masses

In order to map the Higgs mechanism described above to the type IIA side, we need to compute the mass of the massive vector, and describe it in the language of the type IIA string theory.

Let us first recall the mapping in the absence of fluxes between the heterotic string and the type IIA string. On the heterotic side, the $K3$ manifold is taken to be a fibration of T_f^2 over some base B. On the type IIA side we have a Calabi-Yau manifold which is a fibration of some $\tilde{K}3$ over B (where we used fiber-wise the duality between the heterotic string theory on T^4 and the type IIA string theory on $\tilde{K}3$).

The relations between the parameters of the two theories are (denoting the volume of a cycle by V, and not writing down all the numerical constants):

The mapping of the four dimensional Planck scales gives

$$V(K3)V(T^2)/g_h^2 l_h^8 = V(\tilde{K}3)V(B)/g_{\rm II}^2 l_{\rm II}^8. \tag{3.7}$$

For the mapping of the type IIA string to a wrapped heterotic five-brane we have

$$V(T^2)V(T_f^2)/g_h^2 l_h^6 = 1/l_{\rm II}^2. \tag{3.8}$$

The mapping of the heterotic string to a wrapped NS5-brane yields

$$1/l_h^2 = V(\tilde{K}3)/g_{\rm II}^2 l_{\rm II}^6. \tag{3.9}$$

Finally, the integral of the heterotic B-field on the T^2 maps to the integral of the type IIA B-field on some 2-cycle W in $\tilde{K}3$, leading to

$$V(T^2)/l_h^2 = V(W)/l_{\rm II}^2. \tag{3.10}$$

Above we found that on the heterotic side the mass of the vector field that becomes massive after we turn on the flux is

$$m^2 = (f^a)^2 l_h^2/V(T^2)^2. \tag{3.11}$$

Translating this into type IIA string theory using the equations above, we find that the mass can be written as

$$m^2 = (f^a)^2 V(\tilde{K}3)/(V(W)^2 g_{\rm II}^2 l_{\rm II}^2). \tag{3.12}$$

In particular, it involves a negative power of the type IIA string coupling, implying that it is not a perturbative state on the type IIA side. Rather, since its mass is proportional to the D0-brane mass $M_{D0} \simeq 1/g_{\rm II} l_{\rm II}$, it involves when lifted to M-theory some non-trivial momentum on the M-theory circle. Thus, we cannot describe this flux purely in the language of type IIA supergravity (the massive gauge field is too massive to be included in the low energy IIA description). The dual configuration must involve, when lifted to M-theory, non-trivial dependence on the M-theory circle.

3.3 The flux as a monodromy

We claim that the correct description of this flux on the type IIA side is given by the non-trivial fibration of the Calabi-Yau over the M-theory circle, described in the previous section. In order to make this identification more precise, let us move up one dimension, and consider the heterotic string theory on $K3 \times S^1$, which is dual to M-theory on a Calabi-Yau manifold (this is simply the limit of the duality discussed in the previous subsection, when one of the heterotic circles is taken to be large). We will call the coordinate on this circle x^5, and denote the coordinate on the additional circle which we use to go down to four dimensions by x^4 (this may be identified with the z coordinate which we used in section 2).

In the $K3 \times S^1$ compactification, each ten-dimensional gauge field A^a_μ leads to a scalar field A^a_5. One way to describe the flux that we are interested in is by taking this scalar field to have a non-trivial monodromy around the additional circle in the x^4 direction,

$$A^a_5 = cf^a x^4 \Rightarrow A^a_5(x^4 + 2\pi R_4) \simeq A^a_5(x^4) + 2\pi c f^a R_4 \tag{3.13}$$

for some constant c. Note that the low-energy supergravity is invariant under any shift in the scalar field A^a_5; however, in the full heterotic string theory, due to the presence of charged states carrying momentum on the x^5 circle, there is only a discrete periodicity of the field A^a_5. Equation (3.13) may be interpreted as saying that when we go around the x^4 circle, A^a_5 comes back to itself up to a shift by an integer multiple of its period (proportional to f^a).

In this language, we can think of the flux as a special case of a monodromy in the T-duality group. Recall that the heterotic string theory on $K3 \times S^1$ has n U(1) vector fields A^a_μ coming from the ten-dimensional gauge group, and three additional vector fields coming from $g_{\mu 5}$, $B_{\mu 5}$ and the dual of $B_{\mu\nu}$. One combination of the three latter fields is in the graviton multiplet, while the other $n_V^{(5)} = n + 2$ fields are in vector multiplets. Each of the vector multiplets contains a real scalar field; these $n_V^{(5)}$ fields are A^a_5, the radius of the x^5 circle, and the heterotic dilaton, and they span the manifold [22, 24] $\mathrm{SO}(1, n_V^{(5)} - 1)/\mathrm{SO}(n_V^{(5)} - 1) \times \mathbf{R}$. The low-energy supergravity action is invariant under an $\mathrm{SO}(1, n_V^{(5)} - 1) \times \mathrm{SO}(1, 1)$ symmetry, where the first factor rotates the scalars (and all the vector fields except for the dual of $B_{\mu\nu}$), while the second factor shifts the dilaton. In the full heterotic string theory, only an $\mathrm{SO}(1, n_V^{(5)} - 1, \mathbf{Z})$ subgroup of this group is an exact symmetry — this is the T-duality group of the heterotic string on a circle. This group includes in particular the shifts in A^a_5 described in the previous paragraph. Thus, these shifts are a special case of a general $\mathrm{SO}(1, n_V^{(5)} - 1, \mathbf{Z})$ monodromy, where as we go around the circle the theory comes back to itself up to some $\mathrm{SO}(1, n_V^{(5)} - 1, \mathbf{Z})$ transformation.

It is now clear, that in order to map the flux to the M-theory side, we need to consider backgrounds in which M-theory on a Calabi-Yau comes back to itself (as we go around the circle) up to some element of $\mathrm{SO}(1, n_V^{(5)} - 1, \mathbf{Z})$. These are precisely the backgrounds we considered in the previous section, so we claim that these are the correct type II duals of the heterotic compactification with flux. In the next two subsections we will check this proposal in detail, by mapping the four dimensional effective actions of the two theories.

3.4 The low-energy effective action

Let us briefly recall the low energy effective action for heterotic string compactifications on $K3 \times T^2$ with non-trivial background fluxes, which was derived in [17]. In the spirit of the present paper we only focus on the vector multiplets and only review the low energy theory for fluxes of the gauge fields on T^2, as they lead to a non-Abelian gauge group in the effective four-dimensional theory. The main features of the ungauged theory are summarized in appendix B.

The $n_v = n+3$ four dimensional heterotic vector multiplets include the complex scalar fields $x^i = (s, u, t, n^a)$, $a = 4, \ldots, n_v$ which span the symmetric space (B.1), where s denotes the dilaton/axion, t and u are the T^2 moduli and n^a denotes the scalars arising from the

Wilson lines of the original heterotic gauge fields in the T^2 directions. The latter combine with the four-dimensional gauge fields A^a which also originate from the ten-dimensional heterotic gauge fields. From the metric and the B-field we obtain four Kaluza-Klein gauge bosons $A^0, \ldots, A^3$ which play the role of the graviphoton and the superpartners of s, t and u.[12] In the absence of fluxes the gauge group is the Abelian group $[\mathrm{U}(1)]^{(n_v+1)}$.

When we turn on background fluxes of the form

$$\int_{T^2} F^a = f^a \,, \tag{3.14}$$

the four dimensional gauge group becomes non-Abelian (in the sense that different gauge transformations no longer commute), as in the general gauged supergravities discussed in the appendix. Note that this non-Abelian symmetry has nothing to do with the original $E_8 \times E_8$ or SO(32) gauge symmetry in ten dimensions; it involves only fields in the Cartan subgroup of the original gauge group.

The action computed in [17] is[13]

$$S_{\mathrm{het}} = \int \left[\frac{1}{2} R * 1 + \frac{1}{4} I_{IJ} F^I \wedge * F^J + \frac{1}{4} R_{IJ} F^I \wedge F^J - g_{ij} Dx^i \wedge * D\bar{x}^{\bar{j}} - V \right] , \tag{3.15}$$

which slightly differs from the action given in (A.16). The point is that from the heterotic viewpoint a different symplectic basis is more natural. More precisely, the gauge field A_1 is dualized relative to the formalism used in the appendix, which is the one we use for M-theory. In this basis the prepotential $\mathcal{F}$ does not exist but its derivatives are well defined [35 − 37]. So let us carefully go through the terms.

The non-trivial covariant derivatives in (3.15) when we turn on the fluxes are given by

$$\begin{aligned} Dt &= \partial t - \sqrt{2}\, n^a f^a A^1 + f^a A^a \,, \\ Dn^a &= \partial n^a - \frac{1}{\sqrt{2}} f^a (A^0 + u A^1) \,, \end{aligned} \tag{3.16}$$

which, using (A.12), corresponds to the Killing vectors

$$k_0 = \frac{1}{\sqrt{2}} f^a \partial_a \,, \qquad k_1 = \frac{1}{\sqrt{2}} f^a u\, \partial_a + \sqrt{2}\, n^a f^a \partial_t \,, \qquad k_a = -f^a \partial_t \,. \tag{3.17}$$

Finally, the metric g_{ij} in (3.15) is special Kähler and can be derived from (B.8).

As explained in appendix B, the gauge couplings I_{IJ}, R_{IJ}, which are given in (B.7), cannot be derived directly from (A.3). In the ungauged case ($f^a = 0$) one needs to perform an electric-magnetic duality transformation on the symplectic vector $X^I, \mathcal{F}_I$ given by $X^1 \to -\mathcal{F}_1$ and $\mathcal{F}_1 \to X^1$. Using (A.9) this transforms the gauge couplings I_{IJ}, R_{IJ} into a form consistent with (A.2) and (A.3) while the Kähler potential is left invariant. For the gauged case ($f^a \neq 0$) this transformation is not straightforward and generates precisely a term of the form (A.18) as we will see in the next subsection.

[12] The details can be found in reference [17].

[13] Compared to [17] we have rescaled the metric by a factor 1/2 and the gauge fields by a factor $1/\sqrt{2}$ in order to agree with the conventions we use in type IIA compactifications.

The non-Abelian field strengths in the heterotic basis are given by

$$\begin{aligned} F^0 &= dA^0 \ , \\ F^1 &= dA^1 \ , \\ F^2 &= dA^2 + f^a A^a \wedge A^1 \ , \\ F^3 &= dA^3 - f^a A^a \wedge A^0 \ , \\ F^a &= dA^a - f^a A^0 \wedge A^1 \ . \end{aligned} \tag{3.18}$$

The equations can be understood as follows: recall that (when we do not turn on any non-trivial fields) A^0 and A^1 are linear combinations of $g_{\mu 4}$ and $g_{\mu 5}$, while A^2 and A^3 are linear combinations of $B_{\mu 4}$ and $B_{\mu 5}$. The non-Abelian terms in F^2 and F^3 follow from (3.5) when including off-diagonal metric elements in the contractions. The non-Abelian term in F^a arises just from off-diagonal contractions in the standard six dimensional kinetic term of F^a. By comparing with (A.13) we see that the non-vanishing structure constants are

$$f_{a1}^2 = -f_{a0}^3 = f_{01}^a = f^a \ . \tag{3.19}$$

Note that there is a slight subtlety when one takes the Killing vectors as given in (3.17) and checks the consistency of (3.19) with (A.14). The reason is that the structure constants (3.19) correspond to a Lie algebra generated by $(T_0,\ T_1,\ T_2,\ T_3,\ T_a)$ obeying

$$[T_0, T_1] = f^a T_a \ , \qquad [T_0, T_a] = f^a T_3 \ , \qquad [T_a, T_1] = f^a T_2 \ , \tag{3.20}$$

with all the other commutators vanishing. We see that T_2 and T_3 are central elements of the algebra and therefore can consistently be set to zero. This is precisely what happened in our case in that the Killing vectors k_2 and k_3 are vanishing in (3.16), and therefore the last two commutators in (3.20) are zero even though the corresponding structure constants are non-zero. This situation is encountered frequently in gauged supergravities, see for example [38, 39].[14]

Finally, the potential in the action (3.15) is given by the standard formula (A.15) with the Killing vectors (3.17) inserted.

3.5 Comparison to M-theory

In this section we wish to compare the heterotic flux compactification derived in the previous subsections, with the M-theory compactification of the previous section. For this we have to remember that in the ungauged case the map between heterotic and type IIA theories involves the non-trivial symplectic rotation (B.11). On the gauge fields this translates into the map

$$\begin{aligned} A^0_{\text{het}} &\equiv -A^2_{\text{IIA}} \ , \\ A^1_{\text{het}} &\equiv A^0_{\text{IIA}} \ , \end{aligned}$$

[14]We thank Marco Zagermann for educating us on this subject and the referee of this paper for pointing out that for $T_2 = T_3 = 0$ (3.20) is also a nilpotent Lie algebra [30].

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$$\begin{aligned} A^2_{\text{het}} &\equiv A^3_{\text{IIA}} \,, \\ A^3_{\text{het}} &\equiv \tilde{A}^1_{\text{IIA}} \,, \\ A^a_{\text{het}} &\equiv \sqrt{2} A^a_{\text{IIA}} \,, \end{aligned} \tag{3.21}$$

where $\tilde{A}^1$ denotes the electric-magnetic dual of the vector field A^1 which appears in the type IIA compactification.

In order to compare the low-energy effective actions, we need to insert the M_i^j into eq. (2.58) and compare the resulting covariant derivatives and Killing vectors to the heterotic side as given in (3.17). We immediately see that there is no perfect match between all the M-theory parameters and the heterotic fluxes that we discussed thus far, and we will return to this point later.

However, let us first see for which subset of the M-theory torsion parameters, the heterotic flux can be recovered. Indeed, choosing

$$m_2 = m_3 = m_a = m_b^a = 0 \,, \tag{3.22}$$

and leaving only $\tilde{m}_a \neq 0$ in eq. (2.58) results in the non-trivial covariant derivatives

$$\begin{aligned} D_\mu x^3 &= \partial_\mu x^3 + \tilde{m}_a (x^a A^0_\mu - A^a_\mu) \,, \\ D_\mu x^a &= \partial_\mu x^a + \frac{1}{2} \tilde{m}_a (x^2 A^0_\mu - A^2_\mu) \,, \end{aligned} \tag{3.23}$$

or equivalently the Killing vectors

$$k_0^3 = -x^a \tilde{m}_a \,, \qquad k_0^a = -\frac{1}{2} x^2 \tilde{m}_a \,, \qquad 2k_2^a = k_a^3 = \tilde{m}_a \,. \tag{3.24}$$

Comparison with eq. (3.17) together with the identifications (B.10) and (3.21) shows a perfect match if we identify

$$\tilde{m}_a|_{\text{IIA}} = -\sqrt{2} f^a|_{\text{heterotic}} \,. \tag{3.25}$$

We can similarly compare the field strengths. Inserting eq. (3.22) into (2.38) we arrive at

$$\begin{aligned} F^3 &= dA^3 + \tilde{m}_a A^0 \wedge A^a \,, \\ F^a &= dA^a - \frac{1}{2} \tilde{m}_a A^0 \wedge A^2 \,. \end{aligned} \tag{3.26}$$

Comparing with eq. (3.18) using eqs. (3.25) and (3.21) we see that the field strengths F^3 and F^a above precisely correspond to F^2 and F^a on the heterotic side. However, F^1 on the type IIA/M-theory side is Abelian while its correspondent (via (3.21)), F^3, on the heterotic side is non-Abelian. On the other hand the M-theory side has an additional term (the last term in (2.53)) in the low energy effective action. The reason for this mismatch is the fact that the two actions are computed in different symplectic frames. In the ungauged case (i.e. for $\tilde{m}_a = 0$) one easily identifies a symplectic rotation which connects the two frames. In the gauged case (i.e. for $\tilde{m}_a \neq 0$) this is less straightforward and will occupy us for the rest of this section.[15]

[15] The following discussion should be straightforward in the framework of gauged supergravity as given in [40].

Let us first recall that the presence of the last term in eq. (2.53) was due to the fact that the prepotential (2.54) was not invariant under the gauge transformation (2.36). However in the heterotic frame all terms in eq. (3.15) are invariant and this term is absent. For the choice of parameters (3.22) the last term in eq. (2.53) becomes (up to a total derivative)

$$-\frac{1}{2}\tilde{m}_a A^2 \wedge A^a \wedge dA^1 \ . \tag{3.27}$$

In order to have the two sides match we have to exchange the gauge field A^1 with its magnetic dual.[16] This is indeed possible as the gauge field A^1 appears only via its (Abelian) field strength $F^1 = dA^1$ as can be seen from eqs. (2.53) and (3.27). The easiest way to see how to do the dualization is to add a Lagrange multiplier $-\frac{1}{2}F^1 \wedge d\tilde{A}_1$ which enforces the Bianchi identity of F^1, and $\tilde{A}^1$ will become the magnetic dual of the gauge field A^1. The equation of motion for F^1 then reads

$$\frac{1}{2}\,\mathrm{Im}\mathcal{N}_{1J} * F^J + \frac{1}{2}\,\mathrm{Re}\mathcal{N}_{1J}F^J - \frac{1}{2}\tilde{m}^a A^2 \wedge A^a - \frac{1}{2}d\tilde{A}_1 = 0 \ . \tag{3.28}$$

Defining now the magnetic dual field strength G_1 as

$$G_1 = d\tilde{A}_1 + \tilde{m}^a A^2 \wedge A^a \ , \tag{3.29}$$

the equation of motion for F^1 becomes

$$\frac{1}{2}G_1 \ = \ \frac{1}{2}\,\mathrm{Im}\mathcal{N}_{1J} * F^J + \frac{1}{2}\,\mathrm{Re}\mathcal{N}_{1J}F^J \ \equiv \ \frac{\partial \mathcal{L}_{N=2}}{\partial F^1} \ , \tag{3.30}$$

where $\mathcal{L}_{N=2}$ denotes the generic $N=2$ Lagrangian (A.16). This equation is precisely the definition of magnetic dual field strength in $N=2$ supergravities (A.4) and from here on we can apply the general dualization procedure and transform the matrix of gauge couplings $\mathcal{N}$ as in (A.9) with the matrices U, V, Z and W chosen such that $F^1 \to G_1$ in (A.6).

Clearly, now G_1 defined in eq. (3.29) can be mapped to the heterotic field strength F^3 from eq. (3.18), via eqs. (3.21) and (3.25). This ends the proof that the low energy theories obtained from compactifying heterotic strings on $K3 \times T^2$ with fluxes turned on along T^2 and from compactifying M-theory on a seven-dimensional manifold with SU(3) structure with only the fluxes $\tilde{m}_a$ non-vanishing, are indeed the same.

So far we discussed the duality for the parameter choice (3.22). However, our discussion in the previous section makes it clear that all the parameters M^i_j on the M-theory side, which give rise to consistent backgrounds in the full M-theory,[17] correspond to $\mathrm{SO}(1, n_V^{(5)} - 1, \mathbf{Z})$ monodromies, and they can be described by such monodromies on the heterotic side as well. The specific monodromy we discussed above is simple on the heterotic side since it does not involve the metric, but it is just a shift of the Wilson lines $A_5^{(i)}$ around the torus T^n that they live on. Monodromies in an $\mathrm{SO}(n, \mathbf{Z})$ subgroup of $\mathrm{SO}(1, n_V^{(5)} - 1, \mathbf{Z})$ may be identified as $\mathrm{SL}(n, \mathbf{Z})$ transformations on this torus, which mix the various gauge

[16] Recall that already in six dimensions, the duality between heterotic string theory on T^4 and type IIA string theory on $K3$ involves a dualization of the 2-form field.

[17] Namely, e^M must be a member of the discrete U-duality group.

fields and scalars; these were denoted by M_b^a above. Generic monodromies (involving m_2, m_3 and m_a) do not have a purely geometrical description [41]. For instance, the m_a parameters are related by a T-duality (inverting the radius of one of the circles) to the $\tilde{m}_a$ parameters, so they may be viewed as having a variation of the heterotic gauge fields A^a (similar to (3.13)) along the T-dual circle. However, this "T-dual flux" does not have a geometrical description in the original heterotic language. Finally note that a background with $m_2 + m_3 \neq 0$ is not consistent as it involves a twist with an element of $\mathrm{SO}(1,1,\mathbf{Z})$ which is not part of the U-duality group in five dimensions. This can also be seen from the heterotic side as it would make the heterotic dilaton charged, which has not been observed so far in perturbation theory.

4. Conclusions

In this paper we studied M-theory compactifications on seven-dimensional manifolds with SU(3)-structure. Specifically we considered a class of such manifolds which can be seen as Calabi-Yau threefolds fibred over a circle. The fibration structure is determined by a specific twist of the second cohomology of the Calabi-Yau as we go around the circle. The consistency of the procedure requires that a discrete isometry in the Calabi-Yau moduli space exists (which is an element of the U-duality group of M-theory compactified on the Calabi-Yau manifold). This is guaranteed for $K3$-fibered Calabi-Yau manifolds which correspond to backgrounds that are dual to the heterotic string compactified on $K3 \times T^2$.

Since in such compactifications the second cohomology of the Calabi-Yau manifold governs the vector multiplet sector, the twisting leads to a gauged supergravity where a subset of the isometries of the vector multiplet moduli space are promoted to local gauge symmetries. A novel feature is that the Kähler moduli are charged, and not only their axionic superpartners as it usually happens in $N = 2$ string compactifications. Moreover this gauging turns out to be non-Abelian which so far had not been obtained in (smooth) compactifications of type IIA string theory or M-theory.

The fact that this gauging should exist is expected from the heterotic — type IIA duality. In heterotic $N = 2$ backgrounds arising from $K3 \times T^2$ compactifications with specific background fluxes only the vector multiplets get charged and the potential has no dependence on the hypermultiplets. However, viewed from the dual type IIA perspective, the masses of the vector fields contain negative powers of the type IIA string coupling. Therefore, in order to consistently keep such states in the effective theory and at the same time ignore the KK states, one has to make sure that the type IIA string coupling is large relative to the size of the Calabi-Yau manifold. This forced us into the M-theory regime, and indeed the dual of the heterotic backgrounds were found among the M-theory backgrounds described above.

The general twisted compactification on the M-theory side contains additional parameters which do not map to fluxes on the heterotic side. However, since we can interpret all such compactifications as twists of the five dimensional theory (obtained from M-theory on the Calabi-Yau, or equivalently from the heterotic string theory on $K3 \times S^1$) by an element of the heterotic T-duality group, they can all be described as T-folds on the heterotic side.

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It would be interesting to study these backgrounds further; work along these lines is in progress [41].

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A. Vector multiplets coupled to $N = 2$ supergravity

This appendix is a short review of $N = 2$ supergravity in four dimensions [32, 31]. A generic spectrum contains the gravitational multiplet, n_V vector multiplets, n_H hypermultiplets and n_T vector multiplets. In this paper we are interested only in the vector multiplet sector and therefore we discard the hyper- and tensor-multiplets.

The vector multiplets contain n_V complex scalars $x^i, i = 1, \ldots, n_V$, which span a special Kähler manifold $\mathcal{M}_V$. This implies that the Kähler potential K is not an arbitrary real function but is determined in terms of a holomorphic prepotential $\mathcal{F}$ according to [32]

$$K = -\ln\left[i\bar{X}^I(\bar{x})\mathcal{F}_I(X) - iX^I(x)\bar{\mathcal{F}}_I(\bar{X})\right] . \tag{A.1}$$

The $X^I, I = 0, \ldots, n_V$ are $(n_V + 1)$ holomorphic functions of the scalars x^i, and $\mathcal{F}_I$ abbreviates the derivative, i.e. $\mathcal{F}_I \equiv \frac{\partial \mathcal{F}(X)}{\partial X^I}$. Furthermore $\mathcal{F}(X)$ is a homogeneous function of degree 2 in X^I, i.e. $X^I \mathcal{F}_I = 2\mathcal{F}$.

The bosonic part of the (ungauged) $N = 2$ action for vector multiplets is given by

$$S = \int \left[\frac{1}{2}R^*\mathbf{1} - g_{i\bar{j}}dx^i \wedge *d\bar{x}^{\bar{j}} + \frac{1}{4}\,\mathrm{Im}\mathcal{N}_{IJ}F^I \wedge *F^J + \frac{1}{4}\,\mathrm{Re}\mathcal{N}_{IJ}F^I \wedge F^J\right], \tag{A.2}$$

where $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$. In the ungauged case the field strengths are Abelian, $F^I = dA^I$, and the matrix of gauge couplings is given by

$$\mathcal{N}_{IJ} = \bar{\mathcal{F}}_{IJ} + 2i \, \frac{\mathrm{Im}\mathcal{F}_{IK} \mathrm{Im}\mathcal{F}_{JL} X^K X^L}{\mathrm{Im}\mathcal{F}_{LK} X^K X^L} \; . \tag{A.3}$$

The equations of motion of the action (A.2) are invariant under generalized electric-magnetic duality transformations. From (A.2) one derives the equations of motion

$$\frac{\partial \mathcal{L}}{\partial A^I} = \frac{1}{2} dG_I = 0 \, , \qquad G_I \equiv 2 \frac{\partial \mathcal{L}}{\partial F^I} = \mathrm{Re}\mathcal{N}_{IJ} F^J + \mathrm{Im}\mathcal{N}_{IJ} * F^J \, , \tag{A.4}$$

while the Bianchi identities read

$$dF^I = 0 \; . \tag{A.5}$$

These equations are invariant under the generalized duality rotations[18]

$$\begin{aligned} F^I &\to U^I{}_J \, F^J + Z^{IJ} \, G_J \, , \\ G_I &\to V_I{}^J \, G_J + W_{IJ} \, F^J \, , \end{aligned} \tag{A.6}$$

where U, V, W and Z are constant, real, $(n_V + 1) \times (n_V + 1)$ matrices which obey

$$\begin{aligned} U^{\mathrm{T}} V - W^{\mathrm{T}} Z = V^{\mathrm{T}} U - Z^{\mathrm{T}} W &= \mathbf{1} \, , \\ U^{\mathrm{T}} W = W^{\mathrm{T}} U \, , \quad Z^{\mathrm{T}} V &= V^{\mathrm{T}} Z \; . \end{aligned} \tag{A.7}$$

Together they form the $(2n_V + 2) \times (2n_V + 2)$ symplectic matrix

$$\mathcal{O} \; = \begin{pmatrix} U & Z \\ W & V \end{pmatrix} . \tag{A.8}$$

Thus (F^I, G_I) form a $(2n_V + 2)$ symplectic vector. Similarly $(X^I, \mathcal{F}_I)$ enjoy the same transformation properties and transform as a symplectic vector under (A.6). The Kähler potential (A.1) is invariant under this symplectic transformation, while the matrix $\mathcal{N}$ transforms according to

$$\mathcal{N} \to (V\mathcal{N} + W) \, (U + Z\mathcal{N})^{-1} \, . \tag{A.9}$$

The isometries of the scalar manifold $\mathcal{M}_V$ are global invariances of the scalar field sector, which can be "gauged" by mixing them with the local symmetries. These isometries are generated by holomorphic Killing vectors $k_I^i(x)$ via

$$\delta x^i \;=\; \Lambda^I k_I^i(x) \; . \tag{A.10}$$

The $k_I^i(x)$ satisfy the Killing equation which in $N = 2$ supergravity can be solved in terms of a Killing prepotential P_I

$$k_I^i(x) = g^{i\bar{j}} \partial_{\bar{j}} P_I \; . \tag{A.11}$$

[18]This is often stated in terms of the self-dual and anti-self-dual part of the field strength $F^{\pm J}$ and the dual quantities $G_I^+ \equiv \mathcal{N}_{IJ} F^{+J}$, $G_I^- \equiv \bar{\mathcal{N}}_{IJ} F^{-J}$.

Gauging the isometries (A.10) requires the replacement of ordinary derivatives by covariant derivatives in the action (A.2)

$$\partial_\mu x^i \to D_\mu x^i = \partial_\mu x^i - k^i_I A^I_\mu\,, \tag{A.12}$$

and the field strengths take the form

$$F^I = dA^I + f^I_{JK} A^J \wedge A^K\ . \tag{A.13}$$

Consistency requires

$$\big[k_I, k_J\big] \ = \ f^L_{IJ}\ k_L\ , \tag{A.14}$$

where $k_I = k^j_I \partial_j$. Furthermore the potential

$$V = 2e^K X^I \bar{X}^J g_{\bar{\imath}j}\, k^{\bar{\imath}}_I k^j_J \tag{A.15}$$

has to be added to the action in order to preserve supersymmetry.[19] The bosonic part of the action of gauged $N = 2$ supergravity is then given by

$$S = \int \left[\frac{1}{2} R^* \mathbf{1} - g_{i\bar{\jmath}} Dx^i \wedge * D\bar{x}^{\bar{\jmath}} + \frac{1}{4}\,\mathrm{Im}\mathcal{N}_{IJ} F^I \wedge * F^J + \frac{1}{4}\,\mathrm{Re}\mathcal{N}_{IJ} F^I \wedge F^J - V\right]\ . \tag{A.16}$$

The symplectic invariance of the ungauged theory is generically broken since the action now explicitly depends on the gauge potentials A^I through the covariant derivatives Dx^i and the non-Abelian field strengths F^I.

There is yet a further generalization of the above setup which was discussed in [32]. The isometries considered above need not leave the prepotential $\mathcal{F}$ invariant. For example, consider an isometry which leads to a change in the prepotential of the type

$$\delta\mathcal{F} = \Lambda^I C_{IJK} X^J X^K\ , \tag{A.17}$$

for some real parameters C_{IJK}. Obviously, the imaginary part of the second derivative of this variation vanishes. From its definition (A.3) we see that the imaginary part of the gauge coupling matrix Im$\mathcal{N}$ is left invariant. Re$\mathcal{N}$ changes however, and so the action as defined in (A.16) is not invariant. In order to restore gauge invariance the following term has to be added to the action [32]

$$S \to S + \int \frac{1}{3} C_{IJK} A^I \wedge A^J \wedge \left(dA^K - \frac{3}{8} f^K_{\mathrm{LM}} A^L \wedge A^M\right)\ . \tag{A.18}$$

B. The vector multiplet sector of heterotic string compactifications on $K3 \times T^2$

In this appendix we review the structure of the vector multiplet sector of heterotic strings compactified on $K3 \times T^2$, following [17]. For this setup, the vector multiplet sector is

[19] Note the factor 2 in front of the potential compared to [31] which comes from the different normalization which we use in the action (A.16).

directly connected to the T^2 part of the compactification and the $K3$ factor only breaks supersymmetry and may reduce the total number of vector multiplets. Therefore, for our purposes studying the T^2 step will be enough. The initial non-Abelian gauge symmetry of the heterotic string is in general broken spontaneously to the maximal Abelian subgroup and therefore we consider the resulting theory to be $N = 2$ supergravity coupled to an arbitrary number n_v of Abelian vector multiplets.

The vector fields in the vector multiplets have two origins: first they can come from gauge fields in ten dimensions (and their number is arbitrary) and second they arise as KK vector fields on the torus. In the last class we have precisely four vector fields, two from the internal components of the metric — which we denote A^0 and A^1 — and two from the B-field — which we denote A^2 and A^3. One of these vector fields, or some combination of them will be the graviphoton, while the rest will sit in vector multiplets. The vector fields from the first class we denote as A^a $(a = 4, \dots, n_v)$, and they are all part of vector multiplets.

The scalar fields in the vector multiplets span the coset space

$$\mathcal{M}_V = \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \otimes \frac{\mathrm{SO}(2, n_v - 1)}{\mathrm{SO}(2) \times \mathrm{SO}(n_v - 1)} \,. \tag{B.1}$$

The factor $\mathrm{SU}(1,1)/\,\mathrm{U}(1)$ corresponds to the dilaton and its superpartner, the axion dual to the four-dimensional B-field, while the second factor describes the scalar fields coming from the T^2 moduli (including the internal B-field) and from the internal components of the ten-dimensional gauge fields. These fields combine into the complex scalar fields $x^i = (s, u, t, n^a),\ a = 4, \dots, n_v$, with s being the heterotic dilaton

$$s = \frac{a}{2} - \frac{i}{2} e^{-\phi} \,, \tag{B.2}$$

while the rest are given implicitly by

$$\begin{aligned}
A_1^a &= \sqrt{2}\frac{n^a - \bar{n}^a}{u - \bar{u}} \,, & A_2^a &= \sqrt{2}\frac{\bar{u}n^a - u\bar{n}^a}{u - \bar{u}} \,, \\
B_{12} &= \frac{1}{2}\left[(t + \bar{t}) - \frac{(n + \bar{n})^a (n - \bar{n})^a}{u - \bar{u}}\right] \,, \\
\sqrt{G} &= -\frac{i}{2}\left[(t - \bar{t}) - \frac{(n - \bar{n})^a (n - \bar{n})^a}{u - \bar{u}}\right] \,, \\
G_{11} &= \frac{2i}{u - \bar{u}}\sqrt{G} \,, & G_{12} &= i\frac{u + \bar{u}}{u - \bar{u}}\sqrt{G} \,,
\end{aligned} \tag{B.3}$$

where $A^a_{1,2}$ denote the internal components of the gauge fields, B_{12} is the internal B-field, while G_{11}, G_{12} and G stand for the metric on the torus and for its determinant, respectively.

From the T^2 compactification point of view, the dynamics of these fields is naturally described in terms of a $\mathrm{SO}(2, n_v - 1)$ matrix M^{IJ} which is given by

$$M = \begin{pmatrix} G^{-1} & -G^{-1}\hat{B} & -G^{-1}A \\ -\hat{B}^T G^{-1} & G + A^T A + \hat{B}^T G^{-1} \hat{B} & A + \hat{B}^T G^{-1} A \\ -A^T G^{-1} & A^T + A^T G^{-1} \hat{B} & \mathbf{1}_{n_v - 3} + A^T G^{-1} A \end{pmatrix} \,, \tag{B.4}$$

where $\hat{B}_{ij} = B_{ij} + \frac{1}{2} A_i^a A_j^a$ with indices i, j labeling the T^2 directions. The matrix M as defined above leaves invariant the $\mathrm{SO}(2, n_v - 1)$ metric

$$\eta = \begin{pmatrix} 0 & \mathbf{1_2} & 0 \\ \mathbf{1_2} & 0 & 0 \\ 0 & 0 & \mathbf{1_{n_v-3}} \end{pmatrix} \tag{B.5}$$

in that $M^{IJ} \eta_{JK} M^{KL} = \eta^{IL}$. Then, the kinetic terms of the moduli are given by

$$L_{kin} = \partial_\mu M^{IJ} \partial^\mu (M^{-1})_{IJ} \ , \tag{B.6}$$

while the gauge kinetic function takes the form

$$I_{IJ} \equiv \mathrm{Im}\mathcal{N}_{IJ} = \frac{s - \bar{s}}{2i} (M^{-1})_{IJ} \ , \qquad R_{IJ} \equiv \mathrm{Re}\mathcal{N}_{IJ} = -\frac{s + \bar{s}}{2} \eta_{IJ} \ . \tag{B.7}$$

The connection to $N = 2$ supergravity is not obvious in the above formulation. Moreover, it turns out that that the natural symplectic basis in this case is one where no prepotential exists [35−37] and so the formulae of appendix A, and in particular the definition of the gauge coupling matrix (A.3), do not directly apply. However one can explicitly compute (B.6) using (B.4) and (B.3) and show that these kinetic terms can be derived from the Kähler potential

$$K = -\ln \left[i(\bar{s} - s) \left((u - \bar{u})(t - \bar{t}) - (n - \bar{n})^a (n - \bar{n})^a \right) \right] \ . \tag{B.8}$$

Moreover one can show that using the general formalism of [31] the gauge coupling matrix (B.7) can be obtained form the following holomorphic vector

$$\left(X^I \mid F_I \right) = \Big(-u, \ 1, \ t, \ ut - n^a n^a, \ \sqrt{2} n^a \mid -st, \ -s(ut - n^a n^a), \ su, \ -s, \ -\sqrt{2} s n \Big) \ , \tag{B.9}$$

while, obviously, using (A.1) this reproduces the Kähler potential (B.8). Alternatively, we can start from the type IIA prepotential (2.62) with the projective coordinates given by

$$X^0 = 1 \ , \quad X^1 = s \ , \quad X^2 = u \ , \quad X^3 = t \ , \quad X^a = n^a \ . \tag{B.10}$$

Using (A.3), one then computes the gauge coupling matrix $\mathcal{N}$. To go to the heterotic symplectic basis (B.9) we perform the symplectic rotation with the matrices U, V, W, Z in (A.8) given by

$$U = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2}\,\mathbf{1_{n_v-3}} \end{pmatrix} , \qquad V = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}}\mathbf{1_{n_v-3}} \end{pmatrix} \tag{B.11}$$

$$Z = -W = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{0_{n_v-3}} \end{pmatrix} .$$

Note that these matrices are precisely the ones which transform the holomorphic section derived from the prepotential (2.62) and (B.10) into (B.9). Moreover, using the transformation of the gauge coupling matrix (A.9) it is completely straightforward, but a bit tedious, to show that the gauge coupling matrix precisely reproduces (B.7). Finally, let us observe that since the matrices Z and W are non-vanishing this transformation is intrinsically a non-perturbative one in that it exchanges the gauge field A^1 with its magnetic dual, followed by certain relabelings and rescalings.

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